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Hydraulic control of homogeneous shear flows

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If a shear flow of a homogeneous fluid preserves the shape of its velocity profile, a standard formula for the condition for hydraulic control suggests that this is achieved when the depth-averaged flow speed is less than $(gh)^{1/2}$. On the other hand, shallow-water waves have a speed relative to the mean flow of more than $(gh)^{1/2}$, suggesting that information could propagate upstream. This apparent paradox is resolved by showing that the internal stress required to maintain a constant velocity profile depends on flow derivatives along the channel, thus altering the wave speed without introducing damping. By contrast, an inviscid shear flow does not maintain the same profile shape, but it can be shown that long waves are stationary at a position of hydraulic control.

1. Introduction

There are numerous flows of importance in engineering, oceanography and meteorology in which flow speeds are comparable with the speed of waves. One class of such problems, involving flow that is also influenced by topographic features, is often termed ‘hydraulics’ if the flow varies slowly in the downstream direction so that the pressure field can be assumed to be hydrostatic. Engineering examples include the flow of a homogeneous fluid with a free surface (Chow 1959; Henderson 1966), as for flow in rivers and open channels. The theory of this applies also to single-layer ‘reduced gravity’ flows in the ocean and atmosphere. Multi-layer and continuously stratified flows occur in both engineering situations and in the ocean and atmosphere for flow over topographic features (Baines 1995). In these latter geophysical situations the Earth’s rotation may play a significant role (e.g. Whitehead 1998).

In many of these situations it is possible for conditions upstream of a topographic constriction, for a given flow rate, to be independent of changes in downstream conditions. Thus the level of a river upstream of a weir is a function only of the discharge and weir height (Chow 1959). The other way round, the discharge of dense water from one ocean basin to another may be simply determined by the relative height of the top of the dense water upstream and the height (and possibly width) of the sill between the basins. These situations are described as being subject to ‘hydraulic control’.

The most physically convincing argument for hydraulic control comes from showing that solutions to the momentum and continuity equations no longer exist if one tries to, say, reduce the reservoir level further. A second approach is to argue that control occurs when long waves can no longer propagate upstream. A third approach comes

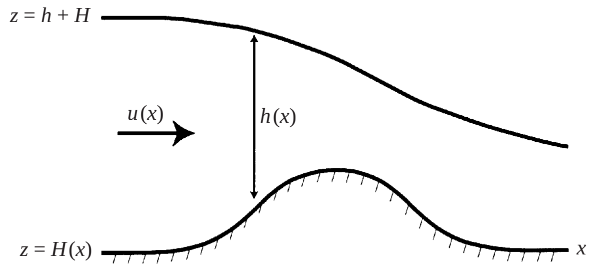

FIGURE 1. A single layer of fluid of depth $h(x)$ flows at speed $u(x)$ over a sill of height $H(x)$.

from examining the mathematical structure of the equations on the assumption that there is an asymmetry in the flow properties through a constriction.

In inviscid situations these three arguments can generally be shown to be equivalent. Incorporating the effects of friction or entrainment can lead to minor modifications if one maintains an assumption of slab-like layers (Pratt 1986; Gerdes, Garrett & Farmer 2002), but further consideration is required if there is frictionally induced shear in the flow. In the engineering literature (Chow 1959; Henderson 1966) this is frequently allowed for with the similarity assumption that the shape of the velocity field is maintained even as its average value changes. It seems not to have been pointed out, however, that this approach leads to the prediction of hydraulic control at a location where long waves can apparently still propagate upstream. Quite apart from conflicting with what had been assumed to be a general result for control, it does seem curious that the level of an upstream basin is apparently unaffected by changes in downstream conditions, even though the information could propagate upstream.

The purpose of this paper is to investigate this apparent paradox. A resolution will be presented in § 3 after a short introduction to the background in § 2. The resolution leads to a question about the nature of control for the open channel flow of an inviscid, rather than frictional, homogeneous fluid that, nonetheless, has a vertically sheared velocity. This will be discussed in § 4.

2. Homogeneous open channel flow

The simplest example to illustrate the basic problem is the hydrostatic (shallow-water) flow of a single inviscid homogeneous layer along a channel of rectangular cross-section and constant width, with bottom elevation $H(x)$. If $u(x)$ is the depth-uniform speed and $h(x)$ the thickness of the layer (figure 1), the momentum and continuity equations are

$$u\frac{\mathrm{d}u}{\mathrm{d}x} + g\frac{\mathrm{d}}{\mathrm{d}x}(h+H) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}(uh) = 0. \tag{2.1}$$

These may be integrated to

$$\tfrac{1}{2}u^2 + gh = g(a-H), \quad uh = Q, \tag{2.2}$$

where a is the height of a reservoir where $u = 0$ and Q is the volume flux per unit width of the channel.

Combining the two equations in (2.2) leads to

$$\frac{1}{2}\frac{Q^2}{h^2} + gh = g(a-H). \tag{2.3}$$

The left-hand side of this has a minimum of $\frac{3}{2}(gQ)^{2/3}$ for $h = (Q^2/g)^{1/3}$ so that, if $H_{\max}$ is the maximum height of a ridge over which the water flows, no solutions are possible if $a < H_{\max} + \frac{3}{2}(Q^2/g)^{1/3}$. The level of the upstream reservoir is thus clearly 'controlled' at this level even if that of a downstream basin is lowered further. At the control section at the ridge crest the Froude number $F = u/(gh)^{1/2}$ is equal to 1, so that long waves, which propagate with speed $(gh)^{1/2}$, cannot propagate upstream. This confirms the equivalence of the first two approaches mentioned earlier.

A more elegant version of (2.3) comes from writing it as

$$\tfrac{1}{2}F^{4/3} + F^{-2/3} = (g/Q^2)^{1/3}(a - H). \tag{2.4}$$

A controlled flow over the top of the ridge becomes asymmetric as the flow switches between the subcritical branch of this, with $F < 1$, to the supercritical branch with $F > 1$.

The third approach (Benton 1954; Armi 1986) comes from recognizing that the determinant of the coefficients of $\mathrm{d}u/\mathrm{d}x$ and $\mathrm{d}h/\mathrm{d}x$ in (2.1) vanishes when $F^2 = 1$, so that a transition from one solution branch to the other, as for asymmetric flows, can only occur at the ridge crest where $\mathrm{d}H/\mathrm{d}x = 0$.

All these approaches can be generalized to more complicated inviscid situations involving, for example, more than one layer (Dalziel 1991) or the effects of rotation (Pratt & Armi 1987). In particular, Gill (1977) recognized that a common property of hydraulic control situations is that the flow can be described in terms of a functional expression connecting a *single* flow variable, h in the example we have given, with the geometric parameters, just H in the simple example above, but easily extended to include, say, the width $W(x)$. This functional expression, say $\mathscr{J}(h; H, W) =$ constant, must be multi-valued, giving more than one solution of h for given values of the geometric parameters H and W. Control also requires the presence of a constriction, in a sense that Gill (1977) extends to be more general than the requirement for a maximum of H in a constant-width channel. As argued by Gill (1977), a transition between branches of the functional occurs where $\partial\mathscr{J}/\partial h = 0$. This implies that $\mathscr{J}(h + \delta h; H, W) =$ constant is also satisfied at the control section, so that a slightly perturbed flow also satisfies the equations for steady flow, implying that waves are stationary there. The conditions of this general approach will turn out to be important when we consider the paradox mentioned in the introduction.

2.1. *Friction*

If a single-layer flow subject to bottom friction is still regarded as being slab-like, the momentum equation becomes

$$u\frac{\mathrm{d}u}{\mathrm{d}x} + g\frac{\mathrm{d}}{\mathrm{d}x}(h + H) = -\frac{C_d u^2}{h}, \tag{2.5}$$

assuming, for example, quadratic bottom friction with drag coefficient C_d. There is no longer a simple functional expression connecting h to H, but there is a simple differential equation equivalent to (2.1),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\frac{Q^2}{h^2} + gh\right) = -g\frac{\mathrm{d}H}{\mathrm{d}x} - \frac{C_d u^2}{h}, \tag{2.6}$$

indicating that bottom friction drives the flow towards criticality, in the same way as a positive bottom slope. The inviscid result is therefore easily extended to reach the conclusion that control, with $F = 1$, can still occur, but is shifted downstream of the ridge crest to a location where $\mathrm{d}H/\mathrm{d}x = -C_d$. This result is implicit in the discussion in some texts (e.g. Henderson 1966) and was also formally derived and applied to oceanographic situations by Pratt (1986) who, along with Bormans & Garrett (1989), also included width variations. Oceanographic applications are assumed to be for 'reduced-gravity' flows of a single layer above or below a deep stagnant layer of a slightly different density. In such cases entrainment into the active layer can also occur; Gerdes *et al.* (2002) showed that this has much the same effect as friction in driving a flow towards criticality on proceeding downstream. Both of these problems could also be analysed using Armi's (1986) approach via the matrix of coefficients of the governing differential equations.

2.2. *Shear flow*

The problem becomes more subtle in a situation where friction leads to a vertical shear as well as an overall retarding force. In this case the pressure may still be hydrostatic if the horizontal scale is much greater than the water depth, but it is no longer appropriate to neglect the advective term involving the vertical shear. The momentum equation is then

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+g\frac{\mathrm{d}}{\mathrm{d}x}(h+H)=\frac{\partial \tau}{\partial z}, \tag{2.7}$$

where τ represents internal frictional forces. The continuity equation is simply $\partial u/\partial x + \partial w/\partial z = 0$.

It is sometimes assumed that the horizontal velocity component u maintains a similar shape, with horizontal variations only of the depth-averaged speed $\overline{u}(x)$. Thus

$$u(x,z)=\overline{u}(x)P(\zeta), \quad \text{where} \quad \zeta=\frac{z-H}{h}, \tag{2.8}$$

and $P(\zeta)$, with $\int_0^1 P\,\mathrm{d}\zeta$, is just a function describing the velocity profile. The continuity equation, along with its vertical integral $\mathrm{d}(h\overline{u})/\mathrm{d}x=0$ and kinematic boundary conditions at $z=H$ and $z=H+h$, then gives

$$w=\overline{u}(x)P(\zeta)\left(\frac{\mathrm{d}H}{\mathrm{d}x}+\zeta\frac{\mathrm{d}h}{\mathrm{d}x}\right) \tag{2.9}$$

and the advective terms in (2.7) can be written simply as $P^2\,\overline{u}\,\mathrm{d}\overline{u}/\mathrm{d}x$. This will have implications for the form of the frictional term on the right-hand side of (2.7), but for now we note that, if this term is ignored, the vertically integrated momentum and continuity equations become

$$M_2\overline{u}\frac{\mathrm{d}\overline{u}}{\mathrm{d}x}+g\frac{\mathrm{d}}{\mathrm{d}x}(h+H)=0, \quad \frac{\mathrm{d}}{\mathrm{d}x}(h\overline{u})=0, \tag{2.10}$$

where $M_2=\int_0^1 P^2\mathrm{d}\zeta$. The first equation here may be derived directly from consideration of the momentum budget for an element of the fluid between the bottom and the free surface and between x and $x+\mathrm{d}x$ if bottom friction is ignored.

The governing equations (2.10) are now similar in form to the governing equations (2.1) for a flow with no shear. Following the procedure used to investigate these, we conclude that control occurs where the Froude number $\overline{u}/(gh)^{1/2}$ based on the depth-averaged current is $M_2^{-1/2}$. This is $\leqslant 1$ as the Cauchy–Schwartz inequality gives $M_2 \geqslant 1$.

An alternative approach, implied by some texts, is to start from an assumption that the energy flux for the layer is conserved with x. This involves $M_3=\int_0^1 P^3\mathrm{d}\zeta$

and leads to a critical Froude number, based on $\overline{u}$ as before, of $M_3^{-1/2}$, a different value from $M_2^{-1/2}$ but also $\leqslant 1$. In the engineering literature (Chow 1959; Henderson 1966) the third moment, M_3 here, is denoted α and termed the energy coefficient or Coriolis coefficient. The second moment M_2 is denoted β and termed the momentum coefficient or Boussinesq coefficient.

The shortcoming of these arguments is, of course, the neglect of the frictional term on the right-hand side of (2.7), even though such a term is required to maintain the profile in the presence of a varying $\overline{u}(x)$. We return to this later, but first investigate whether long waves are arrested at the implied control section.

2.3. *Long-wave speed*

We take $H = 0$ and a basic flow $u(z)$ in water of depth h. We consider infinitesimal perturbations u', w', h' of the velocity components and free surface elevation. The linearized momentum equation is

$$\frac{\partial u'}{\partial t} + u\frac{\partial u'}{\partial x} + w'\frac{\mathrm{d}u}{\mathrm{d}z} + g\frac{\partial h'}{\partial x} = 0. \tag{2.11}$$

Seeking a wave-like solution with the perturbation variables proportional to $\exp\,[\mathrm{i}k(x-ct)]$ and using the continuity equation $\partial u'/\partial x + \partial w'/\partial z = 0$ leads to

$$(c-u)\frac{\partial w'}{\partial z} + w'\frac{\mathrm{d}u}{\mathrm{d}z} = -g\mathrm{i}kh'. \tag{2.12}$$

After dividing this by $(c-u)^2$, the left-hand side may be written as a differential. Integrating the resulting equation vertically and using the linearized free surface kinematic boundary condition $w' = \partial h'/\partial t + u\,\partial h'/\partial x$ at $z = h$ we obtain

$$\int_0^h \frac{g\,\mathrm{d}z}{(c-u)^2} = 1. \tag{2.13}$$

This result is well-established (Thompson 1949; Burns 1953) but has been derived here to illustrate the simplicity of the derivation. It clearly gives the correct limits that $c^2 = gh$ if $u = 0$ and $(c-u)^2 = gh$ if u is independent of depth. Burns (1953) shows that, with the reasonable restrictions that $0 \leqslant u(0) < u(h)$, $\mathrm{d}u/\mathrm{d}z > 0$ and $\mathrm{d}^2u/\mathrm{d}z^2 \leqslant 0$, there are two solutions for c and these do not lie between $u(0)$ and $u(h)$. Moreover, the magnitude of c relative to $\overline{u}$ is greater than $(gh)^{1/2}$. For example, for a small departure from depth-uniform flow, and taking $\overline{u} = 0$ for convenience, we take $u = (gh)^{1/2}\epsilon(z)$ with $|\epsilon| \ll 1$ and $\int_0^h \epsilon\,\mathrm{d}z = 0$. Expansion of the integrand in (2.13) then gives

$$c^2 = gh\left(1 + 3h^{-1}\int_0^h \epsilon^2\,\mathrm{d}z\right) \tag{2.14}$$

as effectively given in Baines (1995, p. 54) where it is derived from the Taylor–Goldstein equation assuming no stratification so that the buoyancy frequency $N = 0$. Thus the magnitude of the speed of long waves is greater than $(gh)^{1/2}$, in agreement with Burns' (1953) general result. The speed of these waves is apparently sufficient to carry information upstream through an alleged control point at which the mean flow is, as shown earlier, less than $(gh)^{1/2}$.

3. Resolving the paradox

For a sheared flow of a homogeneous fluid in an open rectangular channel we have demonstrated a disagreement between two accepted conditions for hydraulic control. One must be wrong.

One possibility for reconciliation might be that the speed of long waves is reduced by the friction that must accompany the maintenance of the shear. We recall, however, that for a slab-like flow the presence of bottom friction merely shifts the control section downstream of a ridge crest (Pratt 1986); the *inviscid* waves are still arrested there as $u = (gh)^{1/2}$ there. Moreover, in that situation it is easy to show (e.g. Wasjowicz 1993) that while the damped waves have a reduction in their phase speed, this reduction decreases to zero as the wavelength decreases. The *group* speed is actually greater than $(gh)^{1/2}$. (If the bottom friction is represented by a linear term $-\lambda u$ in (2.1), the wave group speed is $(gh)^{1/2}[1 - (4ghk^2)^{-1}\lambda^2]^{-1/2}$ for a wavenumber k large enough to make this real.) It seems that bottom friction does not provide a resolution.

On the other hand, we note that Gill's (1977) functional approach can be applied to (2.10) and shows that long-wave solutions of (2.10) are, in fact, stationary at a control point where $\overline{u} = M_2^{-1/2}(gh)^{1/2}$. This is easily seen directly by adding a depth-averaged current perturbation $\overline{u}'$ and an associated perturbation h' to the surface height. Perturbation of the momentum and continuity equations then gives $M_2\overline{u}ik\overline{u}' + gikh' = 0$ and $h'\overline{u} + h\overline{u}' = 0$ for a stationary wave with wavenumber k. Hence $\overline{u} = M_2^{-1/2}(gh)^{1/2}$. Thus the speed of long waves with respect to the mean flow in a system governed by (2.10) is actually $M_2^{-1/2}(gh)^{1/2}$. The difference between this and the speed of inviscid waves must somehow be associated with the internal frictional term on the right-hand side of (2.7).

We first note that the vertical integral of (2.7) is actually

$$M_2\overline{u}\frac{d\overline{u}}{dx} + g\frac{d}{dx}(h + H) = -\frac{\tau_b}{h}, \tag{3.1}$$

where τ_b is the bottom stress. This equation replaces the first equation in (2.10) and can, of course, be obtained directly from a momentum budget. Using also $h\overline{u} = Q$, the volume flux, the problem appears very similar to that for a slab flow with bottom friction, as discussed in §2.1. Control is achieved with $\overline{u} = M_2^{-1/2}(gh)^{1/2}$ and at a location where $dH/dx = -M_2^{-1}C_d$ if we express the bottom drag τ_b as $C_d\overline{u}^2$.

If we now examine the difference between (2.7) (in which, as remarked earlier, the advective terms may be written as $P^2\,\overline{u}\,d\overline{u}/dx$) and its vertical integral (3.1), we require that

$$\frac{\partial \tau}{\partial z} = \left(P^2 - M_2\right)\overline{u}\frac{d\overline{u}}{dx} - \frac{\tau_b}{h}. \tag{3.2}$$

Even in the absence of bottom friction, adding this to the right-hand side of (2.11) must alter the long-wave speed from that given by (2.13) to the value $M_2^{-1/2}(gh)^{1/2}$, as derived above. The reason for this change in long-wave speed, from that for inviscid waves discussed in §2.3, is that the right-hand side of (2.11) acquires an extra 'frictional' term $(P^2 - M_2)\overline{u}\,d\overline{u}'/dx$ which depends on the flow derivative. It thus alternates in sign, providing no net damping for the wave but changing its propagation speed. These conclusions would need to be adjusted if τ_b depended on flow derivatives rather than just the local flow properties. In that case it, too, could be moved to the left-hand side of the momentum equation, with consequent changes in the long-wave speed and control condition.

We conclude that there is really no paradox. For a flow with a fixed shape of its velocity profile, the long waves for the governing equations are indeed arrested at the control, which occurs where the mean flow speed is less than the speed of waves for an inviscid flow. Moreover, as for the slab flow, bottom friction, provided it is dependent on the local flow speed and not its derivative, shifts the control section downstream of the crest of a bump.

4. Control conditions for an inviscid shear flow

While the above discussion seems to have resolved an apparent paradox for a homogeneous shear flow with a fixed shape of the velocity profile, the question remains about conditions for control in a more general situation. One extreme is for a flow with shear but no friction. The momentum equation is then simply

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + g\frac{\mathrm{d}}{\mathrm{d}x}(h+H) = 0. \tag{4.1}$$

Combined with the continuity equation, $\partial u/\partial x + \partial w/\partial z = 0$, (4.1) may be integrated to give

$$\tfrac{1}{2}u^2 + gh = g[a(\psi) - H], \tag{4.2}$$

which is just Bernoulli's equation for a vortical flow, with a a function of the stream function ψ rather than constant as for the unsheared flow.

A first remark is that, if the flow originated in a deep reservoir with a sluggish flow, then (4.2) implies equal acceleration on all streamlines, so that the velocity difference from top to bottom actually diminishes and becomes small. Another way of seeing this is that, for the assumed channel of constant width, the vorticity $\partial u/\partial z$ is conserved along streamlines, again implying a reduction in top-to-bottom velocity difference as the flow thins. Thus a purely inviscid shear flow is unlikely. One has to imagine a flow in which the shear is maintained by friction until the flow approaches a ridge crest in the vicinity of which the friction is unimportant in comparison with the inertial terms.

The problem may then be cast in terms of a functional connection between h and H by writing the continuity equation as

$$h = \int_H^{H+h} \mathrm{d}z = \int_0^Q \frac{\mathrm{d}\psi}{u}, \tag{4.3}$$

where $u = \partial\psi/\partial z$ and Q is the volume flux. This assumes a single-valued connection between ψ and z, equivalent to assuming a unidirectional flow. Combining (4.2) and (4.3) leads to

$$\mathscr{J}(h;H) = \int_0^Q \frac{\mathrm{d}\psi}{(2g)^{1/2}[a(\psi) - (H+h)]^{1/2}} - h = 0. \tag{4.4}$$

This is of the form $\mathscr{J}(h;H) =$ constant required for the applicability of Gill's (1977) arguments. (Pratt & Armi 1987 used a similar approach to derive a condition for control in a rotating flow that was unsheared in the vertical, but sheared across the channel.)

We investigate $\mathscr{J}(h;H)$ by first non-dimensionalizing ψ with Q and a, H, h with $(Q^2/g)^{1/3}$ (but retain the same symbols for the non-dimensionalized variables). The functional equation (4.4) then becomes

$$\mathscr{J}(h;H) = \int_0^1 \frac{\mathrm{d}\psi}{(2)^{1/2}[a(\psi) - (H+h)]^{1/2}} - h = 0. \tag{4.5}$$

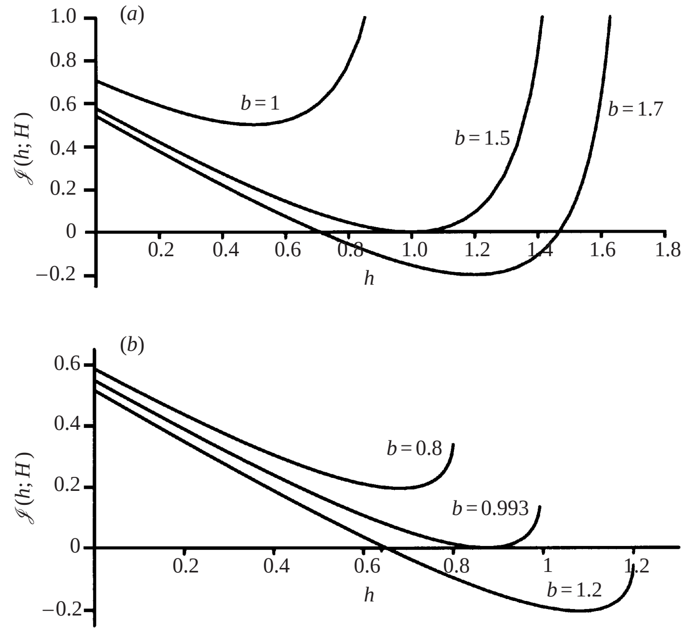

FIGURE 2. (a) The non-dimensionalized functional of (4.5) for constant a, shown for $b = a - H = 1, 3/2, 2$. (b) As in (a), but with $a = a_{min} + 1.5\psi$ and for $b = a_{min} - H = 0.8, 0.993, 1.2$.

For constant a and with $b = a - H$, $\mathscr{J}(h;H) = 2^{-1/2}(b-h)^{-1/2} - h$, as shown in figure 2($a$). It has a minimum of $3/2 - b$ at $h = b - 1/2$, so solutions of (4.5) only exist if $b > 3/2$. Control occurs at the ridge crest if $b = 3/2$ there. This is just another way of looking at the standard problem described in § 2.

The same general shapes occur if a is allowed to vary with ψ, in the sense that the functional starts positive at $h = 0$, then generally decreases to a minimum before increasing as h tends to the minimum value of $a(\psi) - H$, but there is no simple general solution to (4.5). Figure 2(b) shows the functional in (4.5) for various values of $b = a_{min} - H$ with the linear function $a = a_{min} + A\psi$ and $A = 1.5$. Here A is in fact the vertical shear $\partial u/\partial z$. The controlled solution for $A = 1.5$ occurs for $b = 0.993$ and $h = 0.875$; it has non-dimensional speeds of 0.49 and 1.80 at the bottom and free surface respectively. A feature worth noting in figure 2(b) for $b = 1.2$ is that there is no subcritical solution (large h). This is because at a slow average flow, the shear would not be compatible with a positive velocity at the bottom.

As discussed earlier, the functional formalism automatically implies that long waves are arrested at the control section. It is interesting to confirm this directly. Control applies for a solution that satisfies (4.2) and also occurs at the minimum of $\mathscr{J}(h;H)$ regarded as a function of h, i.e. for $\partial\mathscr{J}/\partial h = 0$. Using (4.4) and also applying (4.2), as is appropriate for a solution, leads to the result that

$$\int_0^Q \frac{g\,\mathrm{d}\psi}{u^3} = 1 \tag{4.6}$$

and this may be written as

$$\int_H^{H+h} \frac{g\,\mathrm{d}z}{u^2} = 1. \tag{4.7}$$

This is identical to (2.13) with $c = 0$, implying that long waves are indeed arrested at the control section. Alternatively, we could write $u = \overline{u} + u'(z)$, where $\overline{u'} = 0$ (with an overbar denoting vertical average). Then (4.7) corresponds to (2.13) with $c = -\overline{u}$, so that $\overline{u}$ is equal in magnitude to the speed of long waves for a flow with zero vertical mean.

5. Discussion

We have resolved an apparent paradox concerning the hydraulic control of a frictionally influenced homogeneous fluid with a fixed velocity profile by showing that it does actually fit in with standard approaches if one studies in detail the form of the frictional term required to maintain the shape. The conclusion for practical purposes is that control is indeed achieved with a depth-averaged current speed less than $(gh)^{1/2}$, and at a location downstream of the control for an inviscid flow. We have also shown that an inviscid shear flow is controlled when the depth-averaged current speed equals the long-wave speed, relative to the mean flow, and this speed is greater than $(gh)^{1/2}$. In this situation, however, the velocity difference from bottom to top evolves with distance downstream, decreasing in an accelerating flow. Many real situations will lie between these two limits, in being influenced by friction but not maintaining the shape of the velocity profile. It seems that a key issue is the nature of the frictional forces. If these depend only on local flow properties, then the control section will be shifted, but the speed of inviscid long waves at the control section will still be zero. If, on the other hand, the frictional forces involve along-channel derivatives of the flow variables, as for a fixed velocity profile, then the speed of long waves is changed and the control condition is no longer the same.

We have considered only a rectangular channel of constant width. Allowing for width variations would be trivial, though variations in cross-sectional shape would introduce further difficulties as in inviscid hydraulics. We have also not addressed the effect of stratification, though we note that in the inviscid case Killworth (1992) has shown that control occurs where the long-wave speed vanishes. The viscous case has been investigated by Hogg, Winters & Ivey (2001) who find that, while vortical modes can apparently travel upstream through a constriction in an exchange flow (which is assumed to be controlled), they cannot significantly affect the depth of isopycnals.

The study of hydraulics is surprisingly subtle and it does not seem possible to identify principles that apply to every problem. We hope that the present paper has advanced understanding a little by identifying and resolving what at first appear to be puzzling results for the simple case of a homogeneous fluid.

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